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A note on the limiting procedures for path integrals

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Abstract

Limiting procedures for the Feynman type path integral are considered. The evolution operator is approximated with operators corresponding to the exponential of the Hamiltonian's symbol. The proof of convergence uses Chernoff's theory and its extension on the class of stable operators. The approach is especially adequate for the generalized coherent states path integral, where decomposition of a Hamiltonian into the sums used in the approach based on Trotter's formula may be unnatural.

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1. Introduction

There are several approaches to the rigorous definition of a real-time Feynman path integral [1–12]. One of them is based on a limiting procedure for finite-dimensional approximations. Such a treatment gives a possible way to bypass the problems of measures in the space of continual paths. It is rather useful for many practical purposes. The conventional proof of convergence for the limiting procedure uses Trotter's formula or its generalizations [1–4, 13–16]. It utilizes decomposition of the Hamiltonian into the sums.

In many cases such a decomposition is artificial and may not be adequate for the natural structure of a Hamiltonian. This justifies an interest in approaches which need no decomposition. Decomposition can be discarded within Chernoff's theory [17]. In the present paper, we consider this approach and extend it on the class of stable operators. A treatment is based on the concept of operator's symbols [18–23]. The approach is especially adequate to the generalized coherent states path integrals [24–27].

2. Approximation of the evolution operator

We assume a one-to-one correspondence between operators and their symbols

$$\widehat{A} \xrightarrow{\text{CP}} A.$$

We use (if not stated otherwise) the same letter for the operators and their symbols. It is also assumed that the product of operators $\widehat{A} \cdot \widehat{B}$ corresponds to the convolution of symbols being correspondingly defined as

$$\widehat{A} \cdot \widehat{B} \xrightarrow{\text{CP}} A * B.$$

Such a correspondence often takes place in the theory of pseudo-differential operators and the related context [5, 18–23]. It naturally occurs in the generalized coherent states technique [18]. For convenience, the exact forms of correspondence for some conventional systems of symbols are presented in the appendix.

Let

$$\widehat{U}_t = \exp(-\mathrm{i}t\widehat{H}) \tag{1}$$

be the evolution operator on the time interval $[0, \infty)$ $(\hbar = 1)$.

Let us introduce the operator $\widehat{W}_t$ corresponding to the symbol of the form

$$W_t = \exp(-\mathrm{i}tH), \tag{2}$$

where H is Hamiltonian's symbol. It is assumed that $\widehat{H}$, $\widehat{U}_t$ belongs to the domain and W_t to the range of CP.

Limiting procedures for the Feynman type path integral considered below are based on the formula

$$\widehat{U}_t = \lim_{l\to\infty} \widehat{W}^{\,l}_{\Delta t}, \qquad \Delta t = \frac{t}{l}; \tag{3}$$

the limit is taken in the strong operator topology. In this section, we consider the proof of equation (3).

First, under certain conditions Chernoff's theory [17] can be used. Suppose that operators $\widehat{W}_t$ are strongly continuous and contractive on time interval $t \in [0, \infty)$, that $\widehat{W}_0 = \widehat{1}$ and that $\frac{\mathrm{d}\widehat{W}_t}{\mathrm{d}t}(0) = -\mathrm{i}\widehat{H}$ defines strongly continuous and contractive semigroup equation (1). All operators are considered on some Banach space. Then equation (3) is valid according to Chernoff's theorem [17].

Chernoff's theory and hence equation (3) can be extended on a more general class of stable operators. Let us consider this assertion.

The stability condition for the family of operators $\widehat{W}_{\Delta t}$ on the Banach space is defined by relation [13]

$$\exists\, \alpha, \beta, \tau > 0 \qquad \forall k > 0, \qquad \Delta t \in [0, \tau) \qquad \left\| \widehat{W}^k_{\Delta t} \right\| \leqslant \alpha \cdot \exp(\beta k \Delta t). \tag{4}$$

Let us consider first a special case of stable systems—the so-called proper systems of operators [13].

Let us call bounded operator $\widehat{T}$ a proper one if all its positive powers are uniformly bounded,

$$\exists \alpha \geqslant 1 \qquad \forall k > 0 \qquad \|\widehat{T}^k\| \leqslant \alpha.$$

The system of operators $\widehat{W}_{\Delta t}$ is a proper one if all its operators are proper ones with the same constant α, so that the condition of the form (4) with $\beta = 0$ is valid. The so-called proper semigroup of operators is defined by relations [13]

$$\begin{aligned} &\widehat{S}(t+s) = \widehat{S}(t)\widehat{S}(s), \qquad t, s \geqslant 0, \\ &\lim_{t\to 0} \widehat{S}(t) = \widehat{I}, \\ &\|\widehat{S}(t)\| \leqslant \alpha, \qquad t \geqslant 0. \end{aligned} \tag{5}$$

Chernoff's theory (as well as Trotter's) is based on the Laplace transform relation, connecting the semigroup with its resolvent

$$R_\lambda(\widehat{\Omega}) = (\lambda\widehat{I} - \widehat{\Omega})^{-1} = \int_0^\infty \exp(-\lambda t)\widehat{S}(t)\,\mathrm{d}t, \qquad \lambda > 0,$$

where

$$\widehat{\Omega} = \lim_{t\to 0}\frac{1}{t}(\widehat{S}(t) - \widehat{I})$$

is the infinitesimal generator of the semigroup. The resolvent of the proper semigroups for all $\lambda > 0$ is bounded in norm [13]

$$\|R_\lambda(\widehat{\Omega})\| \leqslant \frac{\alpha}{\lambda}. \tag{6}$$

Let us note that for contraction semigroups $\alpha = 1$ in equation (6).

All proofs of Chernoff's paper can be extended on the proper semigroups with minor changes. In the proof of lemma 1, the bound condition (6) should be used for the norm of resolvent, instead of that with $\psi\Im \leftarrow 1$, valid for contractions. In the proof of the proposition it should be taken into account that as well as for contractions the proper semigroups are uniformly bounded (equation (5)). It is easy to check that lemma 2 can be formulated as follows:

- Let $\widehat{T}$ be a proper operator on Banach space X.
- Then $\exp(t(\widehat{T} - \widehat{I}))$ is a proper semigroup.
- For all $x \in X$ and natural n we have $\|(\exp(n(\widehat{T} - \widehat{I})) - \widehat{T}^n)x\| \leqslant \alpha^2 n^{1/2}\|(\widehat{T} - \widehat{I})x\|$.

Lastly extension of Chernoff's theorem, based on proposition and lemma 2 looks like

- Suppose that $\widehat{W}_t$ is a strongly continuous and proper family of operators on time interval $t \in [0,\infty)$, that $\widehat{W}_0 = \widehat{1}$ and that $\frac{\mathrm{d}\widehat{W}_t}{\mathrm{d}t}(0) = -\mathrm{i}\widehat{H}$ defines the strongly continuous and proper semigroup equation (1).
- Then equation (3) is valid in the strong operator topology.

Extension to the stable operators can be achieved with the trick used in [13]—if $\widehat{W}_{\Delta t}$ is a stable family of operators then the closely related family $\exp(-\beta\Delta t)\cdot\widehat{W}_{\Delta t}$ is a proper one. This proves equation (3) for the class of stable operators.

Let us make some notes.

(I) In connection with the application to the path integral we consider in section 3 and the appendix the case of covariant symbols. For the so-called contravariant symbols convergence of formula similar to equation (3) in the strong operator topology was considered in [22].

(II) Chernoff's treatment essentially uses Trotter's approach [13], based on the relation between a semigroup and its resolvent. It is valid when the spectrum of the generators does not lie on the positive imaginary semiaxis. An alternative treatment can use other estimations of the residual $\widehat{U}^l_{\Delta l} - \widehat{W}^l_{\Delta l}$. There are several works related to this subject in the context of Trotter product formula including the ones with explicit error estimations for some classes of generators [28–31]. The work [29] generalizes the results of Chernoff's theory [17] to the convergence in the operator-norm topology.

Below are given simple sufficient conditions which lead to equation (3) (in operator-norm topology). Let us assume that both families of operators $\widehat{U}_{\Delta t}$, $\widehat{W}_{\Delta t}$ are stable. Taking into account an equality

$$\widehat{U}^l_{\Delta l} - \widehat{W}^l_{\Delta l} = \sum_{k=0}^{l-1} \widehat{U}_{\Delta l}{}^k (\widehat{U}_{\Delta l} - \widehat{W}_{\Delta l}) \widehat{W}^{l-k-1}_{\Delta l}$$

and the stability condition (3) we have

$$\|\widehat{U}^l_{\Delta l} - \widehat{W}^l_{\Delta l}\| \leqslant \sum_{k=0}^{l-1} \|\widehat{U}_{\Delta l}{}^k (\widehat{U}_{\Delta l} - \widehat{W}_{\Delta l}) \widehat{W}^{l-k-1}_{\Delta l}\| \leqslant l \cdot \alpha^2 \exp(\beta t) \|\widehat{U}_{\Delta l} - \widehat{W}_{\Delta l}\|.$$

Performing an exponential expansion the residual $\widehat{U}_{\Delta l} - \widehat{W}_{\Delta l}$ can be written as

$$\widehat{U}_{\Delta l} - \widehat{W}_{\Delta l} = \sum_{k=2}^{\infty} \frac{(\mathrm{i}\Delta t)^k}{k!} (\widehat{H}^k - \overbrace{H^k}) = \Delta t^2 \cdot \widehat{\Omega}_{\Delta t}, \tag{7}$$

where $\overbrace{H^k}$ is an operator with a symbol H^k. The terms of the expansion may be unbounded operators, so the expansion can be used only to find the form of equation (7) in the case of strong operator topology. The expansion should be eliminated in the treatment of the operator-norm convergence. An operator $\widehat{\Omega}_{\Delta t}$ should be directly defined by the left-hand side of equation (7). Obviously, $\widehat{\Omega}_{\Delta t}$ is bounded and normalizable for any $0 < \Delta t < \tau$ (as the left-hand side of equation (7)). Let us assume that for some $0 \leqslant \gamma < 1$ the following value is finite $M = \sup_{\Delta t \in (0,\tau)} \Delta t^{\gamma} \|\widehat{\Omega}_{\Delta t}\| < \infty$. Then we have

$$\|\widehat{U}^l_{\Delta l} - \widehat{W}^l_{\Delta l}\| \leqslant M\alpha^2 \Delta t^{1-\gamma} t \exp(\beta t) \xrightarrow[l\to\infty]{} 0.$$

This confirms operator-norm convergence in equation (3) under made assumptions. A certain limitation of the assumption of finite M is the difficult to verify it [28–31]. For some classes of operator's symbols the derivation of equation (3) on the basis of equation (7) was considered without the proof in [19].

(III) See how Trotter formula follows from equation (3). For example, let us consider a Hamiltonian of the form

$$\widehat{H} = \widehat{T} + \widehat{V} \tag{8}$$

with

$$\widehat{T} = T(\widehat{p}), \qquad \widehat{V} = V(\widehat{q}). \tag{9}$$

This Hamiltonian is naturally to be considered in terms of *qp*-symbols equation (A.17). Using their properties listed in the appendix it can be shown that in this special case we have

$$\widehat{W}_{\Delta t} = \exp(-\mathrm{i}\Delta t \widehat{V}) \cdot \exp(-\mathrm{i}\Delta t \widehat{T}),$$

so equation (3) turns into the conventional Trotter formula.

(IV) In expression (7) for the residual we have $\lim_{\Delta t\to 0} \widehat{\Omega}_{\Delta t} = \frac{1}{2}(\overbrace{H^2} - \widehat{H}^2)$. For the Hamiltonian of the form, equations (8) and (9), the right part of this equality can be expressed through the commutator $\widehat{H}^2 - \overbrace{H^2} = [\widehat{T}, \widehat{V}]$.

(V) Let us consider a simple example of the non-Hermitian Hamiltonian $\widehat{H} = \varepsilon\widehat{q}\widehat{p}$ with the real parameter ε which is not of the type equations (8) and (9). Its qp-symbol is $H = \varepsilon qp$. For simplicity we consider the case of one dimension. Using the formula for the convolution equation (A.19) it can be easily calculated as $(W_{\Delta t} * \psi)(q) = \psi((1-\Delta t\varepsilon)q)$, so $\|\widehat{W}_{\Delta t}\| = (1-\Delta t\varepsilon)^{-1/2}$. Let us restrict operators on the subspace where operator $\mathrm{i}\widehat{H}$ is bounded, so that the semigroup equation (1) is stable. For negative values $\varepsilon < 0$ operators $\widehat{W}_{\Delta t}$ are contractive and equation (3) is valid according to Chernoff's theory. Taking into account that

$$\exists c, \tau > 0 \qquad \forall 0 \leqslant \Delta t < \tau \qquad (1-\Delta t\varepsilon)^{-1/2} \leqslant \exp(c|\varepsilon|\Delta t)$$

operators $\widehat{W}_{\Delta t}$ are stable and equation (3) is valid for all real parameters ε according to the extended theory.

3. A limiting procedure for Feynman-type path integrals

In this section, we consider prelimiting approximations and limiting procedures for a Feynman path integral. The forms of results are well known maybe except for the general case, but using operator's symbols and the coherent states technique together with equation (3) leads to the simple derivation and the proof of convergence valid for a more general class of Hamiltonians. We include in consideration also a relatively less considered case of the fermionic coherent states path integral. A relevant question but from another standpoint for some special cases of operator's symbols not using the generalized coherent states technique was considered in [5].

The existence of some relationship may be noted between definitions of a Feynman path integral based on the limiting procedure and based on the concept of integrators in functional spaces [7–9]. In contrast to the last approach the first one even does not appeal a priory to the functional space of continual paths. At the same time prelimiting approximations have a similar form to approximations, which can be obtained within the integrators technique for some mappings from the space of continual paths into the spaces of finite dimension [7]. This may be treated as implicit evidence of approximations convergence.

3.1. Generalized coherent states path integral

The lth power of operator $\widehat{W}_{\Delta t}$ corresponds to the lth power of convolutions of the operator's symbols $W_{\Delta t}$ equation (A.5). We use notation $W^{*n} = \underbrace{W * \cdots * W}_{n}$. So we have the following representation of the lth power of convolutions in the form of path integral in the space of discrete paths

$$W^{*l}_{\Delta t}(\alpha,\beta) = \exp(-\delta_1\Omega(\alpha,\beta)) \int \exp(S(\tilde{\alpha}))\,\mathrm{d}\tilde{\alpha}, \tag{10}$$

where

$$\begin{gathered}\tilde{\alpha} = (\alpha_0,\ldots,\alpha_l) \text{ is a virtual path, } \alpha = \alpha_l,\ \beta = \alpha_0 \text{ are boundary conditions,}\\ \mathrm{d}\tilde{\alpha} = \prod_{k=1}^{l-1} \mathrm{d}\mu(\alpha_k) \text{ is a measure on the space of virtual paths,}\end{gathered} \tag{11}$$

$$S = \sum_{k=0}^{l-1} \Delta S_k, \qquad \Delta S_k = \delta_1\Omega(\alpha_{k+1},\alpha_k) - \mathrm{i}\Delta t H(\alpha_{k+1},\alpha_k). \tag{12}$$

Equation (12) is the integral sum for the integral

$$S = \int \mathrm{i}d^* \,\mathrm{d}_1\Omega - \mathrm{i}\mathrm{d}t\,H,$$

where $\mathrm{d}_1\Omega(\alpha,\beta) = \frac{\partial\Omega(\alpha,\beta)}{\partial\alpha}\,\mathrm{d}\alpha$ is the differential 1-form on the space Φ^2 and $\mathrm{i}d^*$ is pullback of a diagonal map $\mathrm{i}d : \Phi \to \Delta \subset \Phi^2$, so that $\mathrm{i}d^*\,\mathrm{d}_1\Omega(\alpha,\alpha)$ is the differential 1-form on the space Φ.

As was often noted (for example [19, 27]) there are specific shifts in arguments of functions, entering prelimiting forms of actions (equation (12) and further ones) which gives rise to corresponding infinitesimal time shifts in correct forms of integrals in limit.

For the Hamiltonians satisfying considered in previous section conditions of equation (3) validity we obtain following convergent limiting procedure for the generalized coherent states Feynman path integral representation of the evolution operator's symbol

$$U_t(\alpha,\beta) = \lim_{l\to\infty} W^{*l}_{\Delta t}(\alpha,\beta). \tag{13}$$

So we have the following commutative diagram of the correspondence and Feynman principles

$$\begin{array}{ccc} \widehat{H} & \longrightarrow & \widehat{U}_t \\ \mathrm{CP}\downarrow & & \downarrow\mathrm{CP}, \\ H & \xrightarrow[\mathrm{FP}]{} & U_t \end{array}$$

where both the correspondence principle (CP) and the Feynman principle (FP) depend on the choice of the generalized coherent states system and the arrows should be treated in the same system.

Equations (10)–(13) can represent many other related values, for example, wavefunction's symbol, density operator's symbol, evolution operator's kernel, symbols of functions of Hamiltonian, transition amplitude, scattering cross sections and so on.

3.2. *A case of Kahlerian manifold*

In the case of Kahlerian manifold Φ the general representation, equations (10)–(12), can be written using convolution in the form (A.10) as [26]

$$W^{*l}_{\Delta t}(\alpha^*,\beta) = \exp(-2J\delta^* F(\alpha^*,\beta)) \int \exp(S(\tilde{\alpha}))\,\mathrm{d}\tilde{\alpha}, \tag{14}$$

where $\tilde{\alpha} = (\alpha_0,\ldots,\alpha_l)$ is a virtual path with boundary conditions $\alpha = \alpha_l$, $\beta = \alpha_0$ and

$$\mathrm{d}\tilde{\alpha} = \prod_{k=1}^{l-1} \mathrm{d}\mu(\alpha_k) \tag{15}$$

is a measure on the space of virtual paths with invariant measures equation (A.8),

$$S = \sum_{k=0}^{l-1} \Delta S_k, \qquad \Delta S_k = 2J\delta^* F(\alpha^*_{k+1},\alpha_k) - \mathrm{i}\Delta t\,H(\alpha^*_{k+1},\alpha_k). \tag{16}$$

S is the integral sum for the classical Hamilton–Kahler action

$$S = \int 2J\partial^* F - \mathrm{i}\mathrm{d}t\,H,$$

where $\partial^* F$ is the differential 1-form in the complexification of cotangent bundle on the homogeneous space Φ [23].

Let us note that a convolution written in another form (A.9) leads to a path integral over generalized coherent states of the form presented in [27].

In the case of canonical group generalized coherent states with the Kahlerian phase space $\Phi = C^n$ the representation, equations (14)–(16), can be written using equations (A.7), (A.12) and (A.13) as

$$W_{\Delta t}^{*l}(\alpha^*, \beta) = \exp(-(\alpha - \beta)^*\beta) \int \exp(S(\tilde{\alpha}))\, \mathrm{d}\tilde{\alpha},$$

where

$$\mathrm{d}\tilde{\alpha} = \prod_{k=1}^{l-1} \frac{\mathrm{d}\alpha_k^* \wedge \mathrm{d}\alpha_k}{(2\pi \mathrm{i})^n},$$

$$S = \sum_{k=0}^{l-1} \Delta S_k, \qquad \Delta S_k = \Delta\alpha_k^*\alpha_k - \mathrm{i}\Delta t H(\alpha_{k+1}^*, \alpha_k), \qquad \Delta\alpha_k^* = \alpha_{k+1}^* - \alpha_k^*,$$

S is the integral sum for the classical Hamilton–Kahler action on flat the phase space $\Phi = C^n$

$$S = \int \mathrm{d}\alpha^* \cdot \alpha - \mathrm{i}\mathrm{d}t\ H.$$

3.3. A case of Grassmann algebra

In the case of Grassmann symbols the Feynman path integral representation can be written using equations (A.7), (A.12) and (A.13) as

$$W_{\Delta t}^{*l}(\alpha^*, \beta) = \exp(-(\alpha - \beta)^*\beta) \int \exp(S(\tilde{\alpha}))\, \mathrm{d}\tilde{\alpha}, \tag{17}$$

where $\tilde{\alpha} = (\alpha_0, \ldots, \alpha_l)$ are virtual paths in a set Φ of Grassmann algebra generators with the boundary conditions

$$\alpha = \alpha_l, \qquad \beta = \alpha_0$$

and integration over paths of Grassmann algebra generators (equation (A.14)) [20],

$$\mathrm{d}\tilde{\alpha} = \prod_{k=1}^{l-1} \mathrm{d}\alpha_k^*\, \mathrm{d}\alpha_k.$$

The Grassmann action has the form

$$S = SB - SO,$$

$$SB = \sum_{k=0}^{l-1} \Delta S_k, \qquad \Delta S_k = \Delta\alpha_k^*\alpha_k - \mathrm{i}\Delta t H(\alpha_{k+1}^*, \alpha_k), \qquad \Delta\alpha_k^* = \alpha_{k+1}^* - \alpha_k^*,$$

$$SO = \Delta t \sum_{k=0}^{l-1} H^o(\alpha_{k+1}^*, \alpha_k) \Delta t \sum_{j=0}^{k} H^o(\alpha_{j+1}^*, \alpha_j).$$

The term SO appears for the Hamiltonian with arbitrary Grassmann parity [25] since Grassmann symbols are non-commutative in general case. For arbitrary Grassmann parity elements $A, B \in \Lambda$ one has

$$\exp(A)\exp(B) = \exp(A + B + A^o B^o),$$

where $A = A^e + A^o$ is separation on even—(e) and odd—(o) parts.

SB is the integral sum for the Grassmann action of the form similar to the bosonic case. The additional term *SO* is the integral sum for the integral

$$SO = \int_0^t \mathrm{d}s\, H^o(\alpha_s^*, \alpha_s) \int_0^s \mathrm{d}t\, H^o(\alpha_t^*, \alpha_t).$$

The form of equation (17) for the fermionic Feynman path integral is similar to the bosonic case except for a different form of the Grassmann action. Actions coincide in the case of even Grassmann parity of the Hamiltonian symbol (Hamiltonian even by fermions). In the general case specific additional term appears, which is nonlinear relative to the Hamiltonian and is not local in time [25] (the explicit memory effect in the system).

3.4. A case of *qp*-symbols

The lth power of convolutions of the *qp*-symbols $W_{\Delta t}$ equation (A.17) can be written with equation (A.19) as

$$\begin{aligned}
&W_{\Delta t}^{*l}(q,p) = \int \exp(\mathrm{i}S(\tilde{q},\tilde{p}))\,\mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p}\\
&\tilde{q} = (q_0,\ldots,q_l) \qquad \tilde{p} = (p_0,\ldots,p_{l-1}) \qquad \text{virtual path}\\
&q = q_l = q_0, \qquad p = p_0 \qquad \text{boundary conditions,}\\
&\mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p} = \prod_{k=1}^{l-1} \frac{\mathrm{d}q_k\,\mathrm{d}p_k}{(2\pi)^n},\\
&S = \sum_{k=0}^{l-1} \Delta S_k, \qquad \Delta S_k = \Delta q_k p_k - \Delta t H(q_{k+1}, p_k), \qquad \Delta q_k = q_{k+1} - q_k.
\end{aligned} \tag{18}$$

where S is the integral sum for the classical Hamilton's action.

This leads to the following limiting procedure for the phase-space Feynman path integral representation of the evolution operator's *qp*-symbol

$$U_t(q,p) = \lim_{l\to\infty} W_{\Delta t}^{*l}(q,p).$$

Similar considerations lead to the representation of the wavefunction's $\widehat{U}\psi$ *qp*-symbol equation (A.18)

$$\begin{aligned}
&\langle q|\widehat{U}\psi = \lim_{l\to\infty} \left(W_{\Delta t}^{*l} * \psi\right)(q), \qquad \left(W_{\Delta t}^{*l} * \psi\right)(q) = \int \exp(\mathrm{i}S(\tilde{q},\tilde{p}))\psi(q_0)\,\mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p},\\
&\tilde{q} = (q_0,\ldots,q_l) \qquad \tilde{p} = (p_0,\ldots,p_{l-1})\\
&q = q_l \qquad \text{final condition,}\\
&\mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p} = \prod_{k=0}^{l-1} \frac{\mathrm{d}q_k\,\mathrm{d}p_k}{(2\pi)^n}.
\end{aligned} \tag{19}$$

In the case of the Hamiltonian of the form equations, (8) and (9), with the quadratic term $T(p) = \frac{p^2}{2m}$ the integrals (18) and (19) are the Gaussian integrals over variables $\tilde{p}$. Performing integration, for example, in the case of the integral (19) we obtain the Feynman path integral representation of the wavefunction in the configuration space

$$\langle q|\,\widehat{U}_t\psi = \lim_{l\to\infty}\left(W^{*l}_{\Delta t} * \psi\right)(q), \qquad \left(W^{*l}_{\Delta t} * \psi\right)(q) = \int \exp(\mathrm{i}S(\tilde{q}))\psi(q_0)\,\mathrm{d}\tilde{q},$$

$$\tilde{q} = (q_0, \ldots, q_l), \qquad q = q_l \qquad \text{final condition},$$

$$\mathrm{d}\tilde{q} = \prod_{j=0}^{l-1} C\,\mathrm{d}q_j, \qquad C = \left(\frac{m}{2\pi\, i\,\Delta t}\right)^{\frac{n}{2}},$$

$$S = \sum_{k=0}^{l-1}\Delta S_k, \qquad \Delta S_k = \Delta t\cdot(T\,(mv_k) - V(q_{k+1}))\,, \qquad v_k = \frac{\Delta q_k}{\Delta t},$$

where S is the integral sum for the classical Lagrange action along the virtual path $\tilde{q}$.

4. Conclusions

A finite-dimensional approximation for some types of the path integral considered in this paper uses an approximation of the evolution operator on a time slice with the operator corresponding to the exponential of the Hamiltonian's symbol. The proof of the path integral approximations convergence to the evolution operator symbol uses equation (3), which may be grounded on Chernoff's theory and its extension to the class of stable operators. With this formula Hamiltonian needs no decomposition into any sums as in the case of the Trotter product formula. It is most adequate, for example, to the limiting procedure for the generalized coherent states path integral, where such a decomposition is unnatural.

Appendix

A.1. General case

For convenience, some known facts about the generalized coherent states technique [18] related to this paper are presented here. The system is associated with some irreducible unitary representation of the group G. Generalized coherent states are parameterized by some group homogeneous space Φ

$$|\alpha\rangle \in \Psi, \qquad \alpha \in \Phi,$$

where Ψ is the representation space. It is usually assumed to be normalized as $\langle\alpha \mid \alpha\rangle = 1$.

An essential feature is resolution of unity

$$\widehat{1} = \int |\alpha\rangle\langle\alpha|\,\mathrm{d}\mu(\alpha), \tag{A.1}$$

where $\mathrm{d}\mu(\alpha)$ is the G-invariant measure on Φ.

The so-called covariant symbols [18–23] are defined as follows:

$$A(\alpha,\beta) = \frac{\langle\alpha|\widehat{A}|\beta\rangle}{\langle\alpha|\beta\rangle}. \tag{A.2}$$

Often covariant symbols are defined only on Φ as $A(\alpha,\alpha)$, but we need its extension equation (A.2) on Φ^2. The symbol of the product of operators $\widehat{A}\cdot\widehat{B}$ is the convolution of symbols of the factors, which with equations (A.1) and (A.2) can be written as

$$A * B(\alpha,\beta) = \langle\alpha|\beta\rangle^{-1}\int A(\alpha,\gamma)\cdot B(\gamma,\beta)\langle\alpha\mid\gamma\rangle\langle\gamma\mid\beta\rangle\,\mathrm{d}\mu(\gamma). \tag{A.3}$$

Twice implementing resolution of unity we obtain the form of restoration of the operator by its symbol

$$\widehat{A} = \int |\alpha\rangle A(\alpha, \beta)\langle\alpha|\beta\rangle\langle\beta|\, \mathrm{d}\mu(\alpha)\, \mathrm{d}\mu(\beta). \tag{A.4}$$

Let us write the scalar product of coherent states in an exponential form as $\langle\alpha \mid \beta\rangle = \exp(\Omega(\alpha, \beta))$. Taking into account the normalizing condition of coherent states $\Omega(\alpha, \alpha) = 0$ an exponential form of their scalar product can be written in terms of partial differences with respect to the first argument $\delta_1\Omega(\alpha, \beta) = \Omega(\alpha, \beta) - \Omega(\beta, \beta)$ as

$$\langle\alpha|\beta\rangle = \exp(\delta_1\Omega(\alpha, \beta)),$$

and the convolution equation (A.3) can be written as

$$A * B(\alpha, \beta) = \int A(\alpha, \gamma) \cdot B(\gamma, \beta) \exp(\delta_1\Omega(\alpha, \gamma) + \delta_1\Omega(\gamma, \beta) - \delta_1\Omega(\alpha, \beta))\, \mathrm{d}\mu(\gamma). \tag{A.5}$$

A more symmetric form uses both partial differences

$$\delta_2\Omega(\alpha, \beta) = \Omega(\alpha, \alpha) - \Omega(\alpha, \beta), \qquad \Delta = \delta_1 - \delta_2,$$
$$\langle\alpha \mid \beta\rangle = \exp\left(\tfrac{1}{2}\Delta\Omega(\alpha, \beta)\right),$$

which yields the convolution written as

$$A * B(\alpha, \beta) = \int A(\alpha, \gamma) \cdot B(\gamma, \beta) \exp\left(\frac{1}{2}\Delta\Omega(\alpha, \gamma) + \frac{1}{2}\Delta\Omega(\gamma, \beta) - \frac{1}{2}\Delta\Omega(\alpha, \beta)\right) \mathrm{d}\mu(\gamma). \tag{A.6}$$

A.2. A case of Kahlerian manifold

An important special case is the group homogeneous space Φ having a structure of Kahlerian manifold. In this case, the complex conjugate in the symbol definition (A.2) is used,

$$A(\alpha^*, \beta) = \frac{\langle\alpha|\widehat{A}|\beta\rangle}{\langle\alpha \mid \beta\rangle}. \tag{A.7}$$

For some groups and some of representations [18] the invariant measure in resolution of unity and the scalar product of coherent states can be expressed with the Kahlerian metric potential $F(\alpha^*, \alpha)$ (and its extension $F(\alpha^*, \beta)$) as follows,

$$\begin{aligned} \mathrm{d}\mu(\alpha) &= m(J) \cdot \det\left(\frac{1}{2\pi \mathrm{i}} \frac{\partial^2 F}{\partial\alpha^* \partial\alpha}\right) \mathrm{d}\alpha^* \wedge \mathrm{d}\alpha, \\ \langle\alpha \mid \beta\rangle &= \exp(2J \cdot F(\alpha^*, \beta) - J \cdot F(\alpha^*, \alpha) - J \cdot F(\beta^*, \beta)), \end{aligned} \tag{A.8}$$

where $J = 0, \frac{1}{2}, 1, \ldots$ is the representation index and $m(J) = 2J + 1$.

It is convenient to write the scalar product of coherent states in terms of Kahlerian metric potential differences

$$\delta^* F(\alpha^*, \beta) = F(\alpha^*, \beta) - F(\beta^*, \beta),$$
$$\delta F(\alpha^*, \beta) = F(\alpha^*, \alpha) - F(\alpha^*, \beta),$$
$$\langle\alpha \mid \beta\rangle = \exp(J \cdot \Delta F(\alpha^*, \beta)), \qquad \Delta = \delta^* - \delta.$$

In these terms the convolution equation (A.3) can be written as (a special case of the general form (A.6))

$$\begin{aligned} A * B(\alpha^*, \beta) = \int & A(\alpha, \gamma) \cdot B(\gamma, \beta) \\ & \times \exp(J \cdot (\Delta F(\alpha^*, \gamma) + \Delta F(\gamma^*, \beta) - \Delta F(\alpha^*, \beta)))\, \mathrm{d}\mu(\gamma), \end{aligned} \tag{A.9}$$

or as (a special case of the general form (A.5))

$$A * B(\alpha^*, \beta) = \int A(\alpha, \gamma) \cdot B(\gamma, \beta) \times \exp(2J \cdot (\delta^* F(\alpha^*, \gamma) + \delta^* F(\gamma^*, \beta) - \delta^* F(\alpha^*, \beta)))\, \mathrm{d}\mu(\gamma). \tag{A.10}$$

In the case of canonical group coherent states (sometimes called Glauber or bosonic coherent states) the Kahlerian manifold is flat $\Phi = C^n$ and the Kahlerian metric potential is $F = \alpha^* \cdot \alpha$, so that

$$\mathrm{d}\mu(\alpha) = \frac{\mathrm{d}\alpha^* \wedge \mathrm{d}\alpha}{(2\pi \mathrm{i})^n}, \tag{A.11}$$

$$\langle \alpha \mid \beta \rangle = \exp\left(\alpha^* \beta - \frac{1}{2}(\alpha^* \alpha + \beta^* \beta)\right). \tag{A.12}$$

$$A * B(\alpha^*, \beta) = \int A(\alpha^*, \gamma) \cdot B(\gamma^*, \beta) \cdot \exp((\alpha - \gamma)^* \cdot (\gamma - \beta))\, \mathrm{d}\mu(\gamma). \tag{A.13}$$

A.3. A case of Grassmann algebra

In the case of canonical super-group the generators $\widehat{1}, \widehat{a}^+, \widehat{a}$ satisfy canonical anti-commutation relations. Coherent states (sometimes called fermionic coherent states) $|\alpha\rangle \in \Psi$ are parametrized by generators $\alpha \in \Phi$ ($\dim \Phi = 2n$) of Grassmann algebra Λ ($\dim \Lambda = 2^{2n}$), where n is a number of annihilation operators. The conventional representation space is Fock space F for fermions. Fermionic coherent states introduced in [33] lie in its extension with Grassmann algebra $\Psi = \Lambda \otimes_C F$. It is assumed that involution $*$ is defined on Λ [20], so that generators are split into pair as

$$\alpha = \tfrac{1}{\sqrt{2}}(\xi, i\eta) \qquad \text{and} \qquad \alpha^* = \tfrac{1}{\sqrt{2}}(\xi, -i\eta), \qquad (\dim \xi = \dim \eta = n).$$

Most formulae with fermionic coherent states have the form similar to the bosonic case. The resolution of unity has the form (A.1) where

$$\mathrm{d}\mu(\alpha) = \mathrm{d}\alpha^* \,\mathrm{d}\alpha = \mathrm{i}^n \,\mathrm{d}\xi \,\mathrm{d}\eta, \tag{A.14}$$

with integration over Grassmann algebra generators [20]. Coherent states are eigenfunctions of canonical super-group generators (with provision for duality) with Grassmann eigenvalues

$$\widehat{a}_k|\alpha\rangle = \alpha_k|\alpha\rangle, \qquad \langle\alpha|\widehat{a}_k^+ = \langle\alpha|\alpha_k^*.$$

Covariant symbols, the scalar product and the convolution of symbols are defined by formulae exactly similar to the bosonic case—equations (A.7), (A.12) and (A.13), respectively. Symbols are Grassmann algebra elements.

A.4. A case of qp-symbols

An operator's *qp*-symbols (sometimes called left symbols) [18–22] can be treated in terms resembling definition of the canonical group generalized coherent states symbols. Let

$$\begin{pmatrix} \widehat{q} \\ \widehat{p} \end{pmatrix} = Q \begin{pmatrix} \widehat{a}^+ \\ \widehat{a}^- \end{pmatrix}, \qquad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad Q^{-1} = Q^+$$

be a canonical transform of canonical group generators [18] yielding generators $\widehat{1}, \widehat{q}, \widehat{p}$ satisfying canonical commutation relations in the form

$$\forall j, k \in \{1, \dots, n\} \qquad [\widehat{q}_j, \widehat{p}_k] = \mathrm{i}\delta_{jk}\widehat{1}, \qquad [\widehat{1}, \widehat{q}_j] = [\widehat{1}, \widehat{p}_j] = \widehat{0}.$$

The convensional representation in the space of states Ψ is defined as $\widehat{q} = x$, $\widehat{p} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x}$.

Let us consider two systems of states—one in the canonical group representation space and another in the dual space

$$|p\rangle = \exp(\mathrm{i}xp) \in \Psi, \qquad \langle q| = \delta(x-q) \in \Psi^{+},$$

with the phase-space point $(q, p) \in \Phi = R^{2n}$ as a parameter. It can be called a *qp*-system. Let us note that in contrast to generalized coherent states the space Ψ does not coincide with its dual space Ψ^{+}.

The main features of the *qp*-system resembling the canonical group coherent states system are as follows:

$$\widehat{1} = \int \frac{|p\rangle\langle q|}{\langle q|p\rangle}\frac{\mathrm{d}q\,\mathrm{d}p}{(2\pi)^n} \qquad \text{resolution of unity} \tag{A.15}$$

This follows directly from the Fourier transform properties.

$$\begin{aligned} &\langle q|p\rangle = \exp(\mathrm{i}qp) \qquad \text{normalizing condition.} \\ &\langle q|\widehat{q}_k = \langle q|q_k, \qquad \widehat{p}_k|p\rangle = p_k|p\rangle \end{aligned} \tag{A.16}$$

where *qp*-states are eigenfunctions of canonical group generators (with provision for duality).

Using *qp*-states one can introduce operator's *qp*-symbols as

$$A(q,p) = \frac{\langle q|\widehat{A}|p\rangle}{\langle q|p\rangle}. \tag{A.17}$$

The function's symbol coincides with the function's value

$$\langle q|\psi = \psi(q). \tag{A.18}$$

For the operators being polynomial functions of generators $\widehat{A} = f(\widehat{q}, \widehat{p})$ the *qp*-symbol can be found after operators transform to the so-called *qp*-normal form—all $\widehat{q}$ on the left of all $\widehat{p}$. For normal operator $A(q, p) = f(q, p)$ which is the consequence of equation (A.16).

The *qp*-symbol of the product of operators $\widehat{A}\cdot\widehat{B}$ is the convolution of *qp*-symbols of the factors, which has the form ((A.15) and (A.17) are used)

$$(A * B)(q,p) = \int A(q,p')\cdot B(q',p)\cdot \exp(\mathrm{i}\cdot(q-q')\cdot(p'-p))\frac{\mathrm{d}q'\mathrm{d}p'}{(2\pi)^n}. \tag{A.19}$$

Operators can be restored (with equations (A.15) and (A.17)) by their *qp*-symbols as

$$\widehat{A} = \int \frac{|p\rangle A(q,p')\langle q\mid p'\rangle\langle q'|}{\langle q|p\rangle\langle q'\mid p'\rangle}\frac{\mathrm{d}q\,\mathrm{d}p}{(2\pi)^n}\frac{\mathrm{d}q'\,\mathrm{d}p'}{(2\pi)^n}.$$

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